

FACTORIAL DECAY OF ITERATED ROUGH INTEGRALS

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ABSTRACT. In this complementary note to [1] (arXiv:1501.05641), we provide an alternative proof for the factorial decay estimate of iterated integrals for geometric rough paths without using the neoclassical inequality. This note intends to aid the readers on the proof in [1] which works also for branched rough paths. Just as in [1], the proof here is an extension of Lyons 94' [4] from Young's integration to geometric rough paths.

Let X be a path in a Banach space E and A be a linear map $E \rightarrow L(F, F)$, where F is another Banach space. The controlled differential equation

$$(0.1) \quad dY_t = A(dX_t)(Y_t)$$

has an explicit series expansion of the form

$$(0.2) \quad Y_t = \sum_{k=0}^{\infty} \int_{0 < s_1 < \dots < s_k < t} A(dX_{s_1}) \dots A(dX_{s_k}) Y_0$$

as long as the series converges. As Lyons noted in [5], a first step to make sense of (0.1) is to make sense of the iterated integrals

$$\int_{s < s_1 < \dots < s_n < t} dX_{s_1} \otimes \dots \otimes dX_{s_n}$$

and to prove an estimate for the iterated integral that ensures the series (0.2) converges. The first result in Lyons' original work was in fact aimed to resolve these two questions. To recall Lyons' result, we will use the notation

$$\Delta_n = \{(s_1, s_2, \dots, s_n) : 0 \leq s_1 \leq \dots \leq s_n \leq 1\},$$

and $E^{\otimes 0} = \mathbb{R}$,

$$T^{(n)}(E) = \oplus_{i=0}^n E^{\otimes i}$$

and we will say a map $\mathbb{X} : \Delta_2 \rightarrow T^{(\lfloor p \rfloor)}(E)$ is a multiplicative functional if for all $s \leq u \leq t$,

$$\mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t} = \mathbb{X}_{s,t}.$$

A control is a uniformly continuous function $\omega : \Delta_2 \rightarrow [0, \infty)$ such that for all $s \leq u \leq t$,

$$\omega(s, u) + \omega(u, t) \leq \omega(s, t).$$

Let \mathbb{X}^n denote the projection of \mathbb{X} onto $E^{\otimes n}$. A p -rough path is a multiplicative functional \mathbb{X} such that there exists a constant C (independent of time) and a control ω so that for all $(s, t) \in \Delta_2$,

$$(0.3) \quad \|\mathbb{X}_{s,t}^n\| \leq C\omega(s, t)^{\frac{n}{p}}, \quad \forall n \leq \lfloor p \rfloor.$$

We would like to thank the anonymous referee for [1] for the useful comments.

If (0.3) holds, we say \mathbb{X} is controlled by ω . Here and everywhere below the norm $\|\cdot\|$ can be any norm that is admissible (see Definition 1.25). The readers may wish to just take $\|\cdot\|$ to be the projective norm.

Theorem 1. (*Lyons' Extension Theorem* [5]) *Let $\mathbb{X} : \Delta_2 \rightarrow T^{(\lfloor p \rfloor)}(E)$ be a p -rough path. Suppose further that there exists $\beta \geq p^2(1 + \sum_{r=3}^{\infty} (\frac{2}{r-2})^{\frac{\lfloor p \rfloor + 1}{p}})$ such that*

$$(0.4) \quad \|\mathbb{X}_{s,t}^n\| \leq \frac{1}{\beta(\frac{n}{p})!} \omega(s,t)^{\frac{n}{p}}, \quad \forall n \leq \lfloor p \rfloor,$$

with $(\frac{n}{p})! = \Gamma(\frac{n}{p} + 1)$ and Γ being the gamma function. Then there exists a unique extension of \mathbb{X} to a multiplicative functional, which we will also denote as \mathbb{X} , such that \mathbb{X} is also controlled by ω . Moreover, (0.4) holds for all $n \geq \lfloor p \rfloor + 1$.

The extended multiplicative functional \mathbb{X}^n can be interpreted as the order n iterated integrals of \mathbb{X} . There are several extensions of this estimate for solutions to differential equations, see [7] and [2]. The proof of Theorem 1 uses the ‘‘neoclassical inequality’’ that for all $a, b \geq 0$,

$$\sum_{k=0}^n \frac{a^{(n-k)\frac{1}{p}} b^{\frac{k}{p}}}{(\frac{n-k}{p})! (\frac{k}{p})!} \leq p \frac{(a+b)^{\frac{n}{p}}}{(\frac{n}{p})!}.$$

This neoclassical inequality is due to Hino and Hare [3], although there is a slightly less sharp version of this inequality in Lyons work [5]. The purpose of this article is to give an alternative proof of Lyons' estimate (0.4) without using the neoclassical inequality. By focusing on the simpler case of geometric rough paths, we hope that it will help the readers in understanding the long computations in [1]. We first introduce the notion of factorial control.

Definition 2. Let $m \leq n$. Then we say a uniformly continuous function $R : \Delta_3 \rightarrow [0, \infty)$ is a *factorial control* if $n \geq m$,

1. (Control property for R) for all $u \leq v \leq s \leq t$,

$$R_u^{m,n}(v, s)^{\frac{1}{m}} + R_u^{m,n}(s, t)^{\frac{1}{m}} \leq R_u^{m,n}(v, t)^{\frac{1}{m}}.$$

2. (Decreasing in m) for all $0 \leq k \leq m$,

$$\frac{1}{(n-m)!} R_u^{m,n}(s, t) \leq \frac{c_m}{(n-m+k)!} R_u^{m-k,n}(s, t).$$

3. (R has factorial decay)

$$\frac{1}{(n-m)!} R_u^{m,n}(u, t) \leq \frac{c_m \omega(u, t)^n}{n!}.$$

4. (R dominates binomial sum)

$$\sum_{i=m}^n \frac{\omega(u, s)^{n-i} \omega(s, t)^i}{(m-i)! i!} \leq \frac{1}{(n-m)!} R_u^{m,n}(s, t).$$

5. (Chen's identity for R)

$$\sum_{k=1}^{m-1} R_u^{m-k, n-k}(v, s) \frac{\omega(s, t)^k}{k!} \leq R_u^{m,n}(v, t).$$

We will now construct an example of factorial control. Let ω be a control. Define

$$\rho_u^a(t) = \frac{1}{a} \omega(u, t)^a$$

and

$$S^{(m)}(\rho_u^a(\cdot))_{s,t} = \int_{s < s_1 < \dots < s_m < t} d\rho_u^a(s_1) \dots d\rho_u^a(s_m).$$

Let

$$(0.5) \quad R_u^{m,n}(s, t) = S^{(m)}(\rho_u^{\frac{n}{m}}(\cdot))_{s,t}.$$

Lemma 3. *The function $R_u^{m,n}(s, t)$ defined in (0.9) is a factorial decay estimate.*

Proof. Note that the R function has the explicit representation

$$(0.6) \quad R_u^{m,n}(s, t) = \left(\frac{m}{n}\right)^m \frac{(\omega(u, t)^{\frac{n}{m}} - \omega(u, s)^{\frac{n}{m}})^m}{m!}.$$

This representation gives automatically property 1. for R -function. To show property 2., note that by the inequality that for $a \leq b$ and $\alpha \geq 1$ we have

$$(a - b)^\alpha \leq a^\alpha - b^\alpha.$$

Using this with $\alpha = \frac{m}{m-k}$, we have

$$\begin{aligned} R_u^{m,n}(s, t) &\leq \left(\frac{m}{n}\right)^m \frac{1}{m!} (\omega(u, t)^{\frac{n}{m-k}} - \omega(u, s)^{\frac{n}{m-k}})^{m-k} \\ &= \frac{m^m}{n^k (m-k)^{m-k}} \frac{(m-k)!}{m!} R_u^{m-k,n}(s, t) \\ &\leq \frac{\exp(m)}{n^k} R_u^{m-k,n}(s, t), \end{aligned}$$

where in the final line we used that $m^m/m! \leq \exp(m)$. Therefore,

$$\begin{aligned} \frac{1}{(n-m)!} R_u^{m,n}(s, t) &\leq \frac{\exp(m)}{n^k (n-m)!} R_u^{m-k,n}(s, t) \\ &\leq \frac{\exp(m)}{(n-m+k)!} R_u^{m-k,n}(s, t). \end{aligned}$$

For property 3., we see from the explicit representation of R (0.6) that

$$R_u^{m,n}(u, t) = \left(\frac{m}{n}\right)^m \frac{\omega(u, t)^n}{m!}.$$

As $m^m/m! \leq \exp(m)$, we have

$$\frac{1}{(n-m)!} R_u^{m,n}(u, t) \leq \exp(m) \frac{\omega(u, t)^n}{n!}.$$

We move on to property 4. Applying Taylor's Theorem with integral form remainder to $x \rightarrow \frac{x^n}{n!}$, we have

$$\sum_{i=m}^n \frac{(y-z)^{n-i} (x-y)^i}{(n-i)! i!} = \int_y^x \frac{(a-z)^{n-m} (x-a)^{m-1}}{(n-m)! (m-1)!} da.$$

By reparametrising a as $v \rightarrow z + \omega(u, v)$ and let $x = z + \omega(u, t)$ and $y = z + \omega(u, s)$, we have

$$\begin{aligned} & \sum_{i=m}^n \frac{\omega(u, s)^{n-i} (\omega(u, t) - \omega(u, s))^i}{(n-i)!i!} \\ &= \int_s^t \frac{\omega(u, v)^{n-m} (\omega(u, t) - \omega(u, v))^{m-1}}{(n-m)!(m-1)!} d\omega(u, v). \end{aligned}$$

Therefore, as ω is a control,

$$\begin{aligned} J &:= \sum_{i=m}^n \frac{\omega(u, s)^{n-i} \omega(s, t)^i}{(n-i)!i!} \\ &\leq \sum_{i=m}^n \frac{\omega(u, s)^{n-i} (\omega(u, t) - \omega(u, s))^i}{(n-i)!i!} \\ &= \int_s^t \frac{\omega(u, v)^{n-m} (\omega(u, t) - \omega(u, v))^{m-1}}{(n-m)!(m-1)!} d\omega(u, v) \\ &= \frac{1}{(n-m)!} \int_{s < s_1 < \dots < s_m < t} \omega(u, s_1)^{n-m} d\omega(u, s_1) \dots d\omega(u, s_m). \end{aligned}$$

Note that as $s_1 < \dots < s_m$,

$$\omega(u, s_1)^{n-m} \leq \Pi_{i=1}^m \omega(u, s_i)^{\frac{n-m}{m}}.$$

Therefore,

$$\begin{aligned} J &\leq \frac{1}{(n-m)!} \int_{s < s_1 < \dots < s_m < t} \Pi_{i=1}^m \omega(u, s_i)^{\frac{n-m}{m}} d\omega(u, s_i) \\ &= \frac{1}{(n-m)!} \int_{s < s_1 < \dots < s_m < t} \Pi_{i=1}^m d\rho_u^{\frac{n}{m}}(s_i) \\ &= \frac{1}{(n-m)!} R_u^{m,n}(s, t). \end{aligned}$$

To show property 5, we note that as ω is a control,

$$\begin{aligned} K &:= \sum_{k=1}^{m-1} R_u^{m-k, n-k}(v, s) \frac{\omega(s, t)^k}{k!} \\ &\leq \sum_{k=1}^{m-1} R_u^{m-k, n-k}(v, s) \frac{(\omega(u, t) - \omega(u, s))^k}{k!} \\ &= \sum_{k=1}^{m-1} \int_{v < s_1 < \dots < s_{m-k} < s} \Pi_{i=1}^{m-k} \omega(u, s_i)^{\frac{n-m}{m-k}} d\omega(u, s_1) \dots d\omega(u, s_{m-k}) \\ &\quad \times \int_{s < s_{m-k+1} < \dots < s_m < t} d\omega(u, s_{m-k+1}) \dots d\omega(u, s_m). \end{aligned}$$

Since

$$s_1 < s_2 < \dots < s_{m-k} < s_{m-k+1} < \dots < s_m,$$

we have

$$\Pi_{i=1}^{m-k} \omega(u, s_i)^{\frac{n-m}{m-k}} \leq \Pi_{i=1}^m \omega(s, s_i)^{\frac{n-m}{m}}.$$

Therefore,

$$\begin{aligned}
 K &\leq \sum_{k=1}^{m-1} \int_{v < s_1 < \dots < s_{m-k} < s} \prod_{i=1}^{m-k} \omega(u, s_i)^{\frac{n-m}{m}} d\omega(u, s_i) \\
 &\quad \times \int_{s < s_{m-k+1} < \dots < s_m < t} \prod_{i=m-k+1}^m \omega(u, s_i)^{\frac{n-m}{m}} d\omega(u, s_i) \\
 &= \sum_{k=1}^{m-1} S^{(m-k)}(\rho_u^{\frac{n}{m}})_{v,s} S^{(k)}(\rho_u^{\frac{n}{m}})_{s,t}.
 \end{aligned}$$

By Chen's identity,

$$K \leq S^{(m)}(\rho_u^{\frac{n}{m}})_{v,t} = R^{m,n}$$

□

We will use a trick that first appeared in the work of Young [8]. This involves carefully choosing a sequence of points to be removed from a partition and bounding the change in estimate with each removal. We therefore need the following definition.

Definition 4. Let $\mathbb{X} : \Delta_2 \rightarrow T^{[p]}(E)$ be a multiplicative functional. If $\mathcal{P} = (t_0 < t_1 < \dots < t_r)$ is a partition for $[s, t]$, then we define

$$\mathbb{X}_{s,t}^{n+1,\mathcal{P}} = \sum_{i=0}^{r-1} \sum_{k=1}^{[p]} \mathbb{X}_{s,t_i}^{n+1-k} \otimes \mathbb{X}_{t_i,t_{i+1}}^k.$$

Remark 5. Note that we have $\mathbb{X}_{s,t}^{n+1} = \lim_{\max_i |t_i - t_{i+1}| \rightarrow 0} \mathbb{X}_{s,t}^{n+1,\mathcal{P}}$.

The following algebraic lemma will take care of the algebraic computations in removing points from a partition.

Lemma 6. (Algebraic lemma) Let $\mathbb{X} : \Delta_2 \rightarrow T^{(n)}(E)$ be a multiplicative functional. Then for each t_j in the partition \mathcal{P} of $[s, t]$,

$$\begin{aligned}
 &\sum_{m \geq [p]+1} \mathbb{X}_{u,s}^{n-k} \otimes (\mathbb{X}_{s,t}^{k,\mathcal{P}} - \mathbb{X}_{s,t}^{k,\mathcal{P} \setminus \{t_j\}}) \\
 &= \sum_{k=1}^{[p]} \sum_{l=[p]+1}^{n+1} \mathbb{X}_{u,t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1},t_j}^{l-k} \otimes \mathbb{X}_{t_j,t_{j+1}}^k.
 \end{aligned}$$

Proof. Suppose we define

$$\delta(\mathbb{X}^{n+1}) = \mathbb{X}_{s,t}^{n+1,\mathcal{P}} - \mathbb{X}_{s,t}^{n+1,\mathcal{P} \setminus \{t_j\}}.$$

Note that for any $t_j \in \mathcal{P}$,

$$\begin{aligned}
 \delta(\mathbb{X}^{n+1}) &= \sum_{k=1}^{[p]} \mathbb{X}_{s,t_{j-1}}^{n+1-k} \otimes \mathbb{X}_{t_{j-1},t_j}^k + \sum_{k=1}^{[p]} \mathbb{X}_{s,t_j}^{n+1-k} \otimes \mathbb{X}_{t_j,t_{j+1}}^k \\
 &\quad - \sum_{k=1}^{[p]} \mathbb{X}_{s,t_{j-1}}^{n+1-k} \otimes \mathbb{X}_{t_{j-1},t_{j+1}}^k.
 \end{aligned}$$

Applying the multiplicative property of $\mathbb{X}_{t_{j-1}, t_{j+1}}^k$, we have

$$\begin{aligned} \delta(\mathbb{X}^{n+1}) &= \sum_{k=1}^{\lfloor p \rfloor} \mathbb{X}_{s, t_{j-1}}^{n+1-k} \otimes \mathbb{X}_{t_{j-1}, t_j}^k + \sum_{k=1}^{\lfloor p \rfloor} \mathbb{X}_{s, t_j}^{n+1-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k \\ &\quad - \sum_{k=1}^{\lfloor p \rfloor} \sum_{l=0}^k \mathbb{X}_{s, t_{j-1}}^{n+1-k} \otimes \mathbb{X}_{t_{j-1}, t_j}^{k-l} \otimes \mathbb{X}_{t_j, t_{j+1}}^l. \end{aligned}$$

Note that the term $l = 0$ in the third sum would exactly cancel with the first sum, therefore,

$$\delta(\mathbb{X}^{n+1}) = \sum_{k=1}^{\lfloor p \rfloor} \mathbb{X}_{s, t_j}^{n+1-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k - \sum_{l=1}^{\lfloor p \rfloor} \sum_{k=l}^{\lfloor p \rfloor} \mathbb{X}_{s, t_{j-1}}^{n+1-k} \otimes \mathbb{X}_{t_{j-1}, t_j}^{k-l} \otimes \mathbb{X}_{t_j, t_{j+1}}^l.$$

By renaming variable l as k , and vice-versa, in the second sum, we have

$$\begin{aligned} \delta(\mathbb{X}^{n+1}) &= \sum_{k=1}^{\lfloor p \rfloor} \mathbb{X}_{s, t_j}^{n+1-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k - \sum_{k=1}^{\lfloor p \rfloor} \sum_{l=k}^{\lfloor p \rfloor} \mathbb{X}_{s, t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1}, t_j}^{l-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k \\ &= \sum_{k=1}^{\lfloor p \rfloor} \left(\mathbb{X}_{s, t_j}^{n+1-k} - \sum_{l=k}^{\lfloor p \rfloor} \mathbb{X}_{s, t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1}, t_j}^{l-k} \right) \otimes \mathbb{X}_{t_j, t_{j+1}}^k \\ (0.7) \quad &= \sum_{k=1}^{\lfloor p \rfloor} \sum_{l=\lfloor p \rfloor+1}^{n+1} \mathbb{X}_{s, t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1}, t_j}^{l-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k. \end{aligned}$$

Now by (0.7), reordering the sum and apply the multiplicative property once again, we have

$$\begin{aligned} &\sum_{m=\lfloor p \rfloor+1}^{n+1} \mathbb{X}_{u, s}^{n+1-m} \otimes \delta(\mathbb{X}^m) \\ &= \sum_{m=\lfloor p \rfloor+1}^{n+1} \sum_{k=1}^{\lfloor p \rfloor} \sum_{l=\lfloor p \rfloor+1}^m \mathbb{X}_{u, s}^{n+1-m} \otimes \mathbb{X}_{s, t_{j-1}}^{m-l} \otimes \mathbb{X}_{t_{j-1}, t_j}^{l-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k \\ &= \sum_{l=\lfloor p \rfloor+1}^{n+1} \sum_{k=1}^{\lfloor p \rfloor} \sum_{m=l}^{n+1} \mathbb{X}_{u, s}^{n+1-m} \otimes \mathbb{X}_{s, t_{j-1}}^{m-l} \otimes \mathbb{X}_{t_{j-1}, t_j}^{l-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k \\ &= \sum_{k=1}^{\lfloor p \rfloor} \sum_{l=\lfloor p \rfloor+1}^{n+1} \mathbb{X}_{u, t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1}, t_j}^{l-k} \otimes \mathbb{X}_{t_j, t_{j+1}}^k. \end{aligned}$$

□

We now prove our key proposition that will take us within a short reach of our desired factorial decay estimate.

Proposition 7. *Let ω be a control and let R be a corresponding factorial control. Let $\mathbb{X} : \Delta_2 \rightarrow T^{\lfloor p \rfloor}(E)$ be a p -rough path controlled by ω , more precisely, we assume there exists β such that*

$$\beta \geq [p]^{1-\frac{1}{p}} \zeta\left(\frac{\lfloor p \rfloor + 1}{p}\right) 2^{\frac{\lfloor p \rfloor + 1}{p}} \left[\exp(\lfloor p \rfloor + 1) + [p]^{1-\frac{1}{p}} c_p \right]$$

where c_p is defined in Definition 2 and

$$(0.8) \quad \|\mathbb{X}_{s,t}^k\| \leq \frac{\omega(s,t)^{k/p}}{\beta(k!)^{\frac{1}{p}}} \forall 1 \leq k \leq [p],$$

then for all $m \geq [p]$,

$$(0.9) \quad \|\mathbb{X}_{u,t}^m - \sum_{k \leq [p]} \mathbb{X}_{u,s}^{m-k} \otimes \mathbb{X}_{s,t}^k\| \leq \frac{1}{\beta(m - [p] - 1)!^{\frac{1}{p}}} R_u^{[p]+1,m}(s,t)^{\frac{1}{p}}.$$

Remark 8. Before embarking on the proof, we first show that whenever (0.9) and (0.8) holds, we have for all $k \leq [p] + 1$,

$$\left\| \sum_{i \geq [p]+1-k} \mathbb{X}_{u,s}^{m-i} \otimes \mathbb{X}_{s,t}^i \right\| \leq C_p \beta^{-1} R_u^{[p]+1-k,m}(s,t)^{\frac{1}{p}},$$

for some constant C_p depending only on p . First note that by putting $s = u$ in (0.9) and uses property 3. in Definition 2 that R has factorial decay, we have

$$\begin{aligned} \|\mathbb{X}_{u,t}^m\| &\leq \frac{1}{\beta(m - ([p] + 1))!^{\frac{1}{p}}} R_u^{[p]+1,m}(u,t)^{\frac{1}{p}} \\ &\leq \frac{c_p \omega(u,t)^{m/p}}{\beta(m!)^{1/p}}. \end{aligned}$$

Therefore, let $\tilde{c}_p = [p]^{1-\frac{1}{p}} c_p$ and using property 4. in Definition 2 (R dominates binomial sum),

$$\begin{aligned} \left\| \sum_{i=[p]+1-k}^{[p]} \mathbb{X}_{u,s}^{m-i} \otimes \mathbb{X}_{s,t}^i \right\| &\leq \beta^{-2} c_p \sum_{i=[p]+1-k}^{[p]} \frac{\omega(u,s)^{(m-i)/p} \omega(s,t)^{i/p}}{(m-i)!^{\frac{1}{p}} (i!)^{\frac{1}{p}}} \\ &\leq \beta^{-2} \tilde{c}_p \left(\sum_{i=[p]+1-k}^{[p]} \frac{\omega(u,s)^{m-i} \omega(s,t)^i}{(m-i)! i!} \right)^{\frac{1}{p}} \\ &\leq \beta^{-2} \tilde{c}_p \frac{1}{(m+k-[p]-1)!^{\frac{1}{p}}} R_u^{[p]+1-k,m}(s,t)^{\frac{1}{p}}. \end{aligned}$$

Therefore, by induction

$$\begin{aligned} \left\| \sum_{i=[p]+1-k}^m \mathbb{X}_{u,s}^{m-i} \otimes \mathbb{X}_{s,t}^i \right\| &\leq \beta^{-1} \frac{1}{(m-[p]-1)!^{\frac{1}{p}}} R_u^{[p]+1,m}(s,t)^{\frac{1}{p}} \\ &\quad + \beta^{-2} \tilde{c}_p \frac{1}{(m+k-[p]-1)!^{\frac{1}{p}}} R_u^{[p]+1-k,m}(s,t)^{\frac{1}{p}} \\ &\leq \beta^{-1} C_p \frac{1}{(m+k-[p]-1)!^{\frac{1}{p}}} R_u^{[p]+1-k,m}(s,t)^{\frac{1}{p}}, \end{aligned}$$

where

$$C_p = \exp([p] + 1) + [p]^{1-\frac{1}{p}} c_p.$$

Proof. We shall prove the proposition by induction. The base induction step is trivially true since the left hand side is zero. Assume that (0.9) holds for all $m \leq n$.

By the Algebraic Lemma 6,

$$\begin{aligned}
 I &:= \left\| \sum_{k \geq [p]+1} \mathbb{X}_{u,s}^{n+1-k} \otimes (\mathbb{X}_{s,t}^{k,\mathcal{P}} - \mathbb{X}_{s,t}^{k,\mathcal{P} \setminus \{t_j\}}) \right\| \\
 &= \left\| \sum_{k=1}^{[p]} \sum_{l=[p]+1}^{n+1} \mathbb{X}_{u,t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1},t_j}^{l-k} \otimes \mathbb{X}_{t_j,t_{j+1}}^k \right\| \\
 &\leq \sum_{k=1}^{[p]} \left\| \sum_{l=[p]+1}^{n+1} \mathbb{X}_{u,t_{j-1}}^{n+1-l} \otimes \mathbb{X}_{t_{j-1},t_j}^{l-k} \right\| \left\| \mathbb{X}_{t_j,t_{j+1}}^k \right\| \\
 &= \sum_{k=1}^{[p]} \left\| \sum_{i=[p]+1-k}^{n+1-k} \mathbb{X}_{u,t_{j-1}}^{n+1-k-i} \otimes \mathbb{X}_{t_{j-1},t_j}^i \right\| \left\| \mathbb{X}_{t_j,t_{j+1}}^k \right\|.
 \end{aligned}$$

By (0.8) and Remark 8,

$$\begin{aligned}
 I &\leq \frac{C_p}{(n - [p])!^{\frac{1}{p}}} \sum_{k=1}^{[p]} \frac{1}{\beta} R_u^{[p]+1-k, n+1-k}(t_{j-1}, t_j)^{\frac{1}{p}} \frac{\omega(t_j, t_{j+1})^{k/p}}{\beta(k!)^{1/p}} \\
 &\leq \frac{[p]^{1-\frac{1}{p}} C_p}{\beta^2(n - [p])!^{\frac{1}{p}}} \left(\sum_{k=1}^{[p]} R_u^{[p]+1-k, n+1-k}(t_{j-1}, t_j) \frac{\omega(t_j, t_{j+1})^k}{k!} \right)^{\frac{1}{p}}.
 \end{aligned}$$

It is here that we use Chen's identity for R function (Property 5 in Definition 2) to obtain that

$$I \leq \frac{[p]^{1-\frac{1}{p}} C_p}{\beta^2(n - [p])!^{\frac{1}{p}}} R_u^{[p]+1, n+1}(t_{j-1}, t_{j+1})^{\frac{1}{p}}.$$

Since by the control property of factorial control (property 1. in Definition 2),

$$\sum_{i=1}^{r-1} R_u^{[p]+1, n+1}(t_{i-1}, t_{i+1})^{\frac{1}{[p]+1}} \leq R_u^{[p]+1, n+1}(s, t)^{\frac{1}{[p]+1}},$$

there exists a j such that

$$R_u^{[p]+1, n+1}(t_{j-1}, t_{j+1})^{\frac{1}{[p]+1}} \leq \frac{1}{r-1} R_u^{[p]+1, n+1}(s, t)^{\frac{1}{[p]+1}}.$$

Again by the control property of factorial control (property 1. in Definition 2),

$$R_u^{[p]+1, n+1}(t_{j-1}, t_{j+1}) \leq R_u^{[p]+1, n+1}(s, t).$$

Therefore,

$$R_u^{[p]+1, n+1}(t_{j-1}, t_{j+1})^{\frac{1}{[p]+1}} \leq \left(\frac{2}{r-1} \wedge 1 \right) R_u^{[p]+1, n+1}(s, t)^{\frac{1}{[p]+1}}.$$

This gives us that

$$I \leq \frac{[p]^{1-\frac{1}{p}} C_p}{\beta^2(n - [p])!^{\frac{1}{p}}} \left(\frac{2}{r-1} \wedge 1 \right)^{\frac{[p]+1}{p}} R_u^{[p]+1, n+1}(s, t)^{\frac{1}{p}}.$$

By successively removing points from the partition \mathcal{P} , we have that

$$\begin{aligned} & \left\| \sum_{k \geq [p]+1} \mathbb{X}_{u,s}^{n+1-k} \otimes \mathbb{X}_{s,t}^{k,\mathcal{P}} \right\| \\ &= \left\| \sum_{k \geq [p]+1} \mathbb{X}_{u,s}^{n+1-k} \otimes (\mathbb{X}_{s,t}^{k,\mathcal{P}} - \mathbb{X}_{s,t}^{k,\{s,t\}}) \right\| \\ &\leq \frac{[p]^{1-\frac{1}{p}} C_p 2^{\frac{[p]+1}{p}}}{\beta^2(n-[p])!^{\frac{1}{p}}} \zeta\left(\frac{[p]+1}{p}\right) R_u^{[p]+1,n+1}(s,t)^{\frac{1}{p}}. \end{aligned}$$

We may now take $\beta \geq [p]^{1-\frac{1}{p}} \zeta\left(\frac{[p]+1}{p}\right) C_p 2^{\frac{[p]+1}{p}}$ and take the partition size $|\mathcal{P}| \rightarrow 0$, which gives that

$$\left\| \sum_{k \geq [p]+1} \mathbb{X}_{u,s}^{n+1-k} \otimes \mathbb{X}_{s,t}^k \right\| \leq \frac{1}{\beta(n-[p])!^{\frac{1}{p}}} R_u^{[p]+1,n+1}(s,t)^{\frac{1}{p}}.$$

□

Proposition 9. (*Lyons' factorial decay estimate* [5]) *Let $\mathbb{X} : \Delta_2 \rightarrow T^{([p])}(E)$ be a p -rough path controlled by ω , or more precisely, there exists*

$$\beta \geq [p]^{1-\frac{1}{p}} \zeta\left(\frac{[p]+1}{p}\right) 2^{\frac{[p]+1}{p}} [\exp([p]+1) + [p]^{1-\frac{1}{p}} c_p]$$

such that

$$(0.10) \quad \|\mathbb{X}_{s,t}^k\| \leq \frac{\omega(s,t)^{k/p}}{\beta(k!)^{\frac{1}{p}}} \quad \forall 1 \leq k \leq [p],$$

then for all $m \geq [p] + 1$,

$$\|\mathbb{X}_{s,t}^k\| \leq \frac{c_p \omega(s,t)^{k/p}}{\beta(k!)^{\frac{1}{p}}},$$

where the constant c_p depends only on p and is defined in Definition 2.

Proof. By Proposition 0.10 with $u = s$ we have for all $m \geq [p] + 1$

$$\begin{aligned} \|\mathbb{X}_{u,t}^m\| &\leq \frac{1}{\beta(m-[p]-1)!^{\frac{1}{p}}} R_u^{[p]+1,m}(u,t)^{\frac{1}{p}} \\ &\leq \frac{c_p \omega(u,t)^{\frac{m}{p}}}{\beta(m!)^{\frac{1}{p}}}. \end{aligned}$$

□

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